



Exact boundary conditions for the initial value problem of convex conservation laws [☆]

Zhen-huan Teng ^{*}

LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, China

ARTICLE INFO

Article history:

Received 4 October 2009

Received in revised form 18 January 2010

Accepted 20 January 2010

Available online 28 January 2010

Keywords:

Exact boundary conditions

Artificial boundary conditions

Convex conservation laws

Burgers' (inviscid) equation

Monotone difference schemes

ABSTRACT

The initial value problem of convex conservation laws, which includes the famous Burgers' (inviscid) equation, plays an important role not only in theoretical analysis for conservation laws, but also in numerical computations for various numerical methods. For example, the initial value problem of the Burgers' equation is one of the most popular benchmarks in testing various numerical methods. But in all the numerical tests the initial data have to be assumed that they are either periodic or having a compact support, so that periodic boundary conditions at the periodic boundaries or two constant boundary conditions at two far apart spatial artificial boundaries can be used in practical computations. In this paper for the initial value problem with any initial data we propose exact boundary conditions at two spatial artificial boundaries, which contain a finite computational domain, by using the Lax's exact formulas for the convex conservation laws. The well-posedness of the initial-boundary problem is discussed and the finite difference schemes applied to the artificial boundary problems are described. Numerical tests with the proposed artificial boundary conditions are carried out by using the Lax–Friedrichs monotone difference schemes.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

In this paper we propose exact artificial boundary conditions at two artificial boundaries, which contain a finite computational domain, for the initial value problem of convex conservations:

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0 \quad \text{for } (x, t) \in \mathbf{R} \times \mathbf{R}^+, \quad (1.1)$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \mathbf{R}, \quad (1.2)$$

where the flux $f(u)$ satisfies the convex condition:

$$f''(u) > 0 \quad \text{for } u \in [\inf_x u_0(x), \sup_x u_0(x)]. \quad (1.3)$$

The initial value problem of convex conservation laws, which includes the famous Burgers' (inviscid) equation,

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} u^2 = 0$$

[☆] This work was partially supported by the National Science Foundation of China under Grant No. 10771008.

^{*} Tel.: +86 10 62752699; fax: +86 10 62751801.

E-mail address: tengzh@math.pku.edu.cn

plays an important role not only in theoretical analysis for conservation laws, but also in numerical computations for various numerical methods. For example, the initial value problem of the Burgers' equation is one of the most popular benchmarks in testing various numerical methods, see for example [2,3,5,7,16]. But in all the numerical tests the initial data have to be assumed that they are either periodic or having a compact support, here the compact support means that $u_0(x) = \text{const.}$ for $|x| \gg 1$, so that periodic boundary conditions at the periodic boundaries or two constant boundary conditions at two far apart spatial artificial boundaries can be used in practical computations. In this paper for the initial value problems with any initial data we propose exact boundary conditions at two spatial artificial boundaries, which contain a bounded computational domain, by using the Lax's exact formulas for convex conservation laws. The well-posedness of the artificial boundary problems is discussed and the monotone difference schemes applied to the artificial boundary problems are described. Computing the entropy solution numerically by using the explicit formulas is addressed. Numerical tests with the artificial boundary conditions are carried out by using the Lax–Friedrichs monotone difference schemes.

2. Well-posedness for the initial-boundary value problem of scalar conservation laws

We consider the following initial-boundary value problem

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0 \quad \text{for } (x, t) \in (x_-, x_+) \times \mathbf{R}^+, \tag{2.1}$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in (x_-, x_+), \tag{2.2}$$

$$u(x_-, t) = u_-(t); \quad u(x_+, t) = u_+(t) \quad \text{for } t \in \mathbf{R}^+, \tag{2.3}$$

where x_- and x_+ are two spatial boundaries with $x_- < x_+$ and $u_-(t)$ and $u_+(t)$ are two boundary data.

It is well known that the initial-boundary value problem (2.1)–(2.3) is usually not well posed if the boundary values (2.3) are assumed in the strong sense. Following the pioneering work by Bardos et al. [1] the boundary conditions (2.3) have to be satisfied in the following (weak) sense:

$$\pm(\text{sgn}(u(x_{\pm}, t) - k) - \text{sgn}(u_{\pm}(t) - k))(f(u(x_{\pm}, t)) - f(k)) \leq 0 \quad \forall k \in \mathbf{R}, \tag{2.3'}$$

where the function of sgn is defined by

$$\text{sgn}(x) = \begin{cases} x/|x| & x \neq 0, \\ 0 & x = 0. \end{cases} \tag{2.4}$$

We note that when f is linear, then (2.3') requires u to be equal to the given boundary data $u_{\pm}(t)$ only on the inflow boundaries (where $\pm f'|_{x_{\pm}} \leq 0$) but does not impose any boundary condition on the outflow boundaries (where $\pm f'|_{x_{\pm}} \geq 0$).

More generally, Bardos et al. [1] proved that viscous solutions u^ε satisfying the following initial-boundary parabolic equation with a small viscosity parameter $\varepsilon > 0$

$$\frac{\partial}{\partial t} u^\varepsilon + \frac{\partial}{\partial x} f(u^\varepsilon) = \varepsilon \frac{\partial^2}{\partial x^2} u^\varepsilon \quad \text{for } (x, t) \in (x_-, x_+) \times \mathbf{R}^+, \tag{2.5}$$

$$u^\varepsilon(x, 0) = u_0(x) \quad \text{for } x \in (x_-, x_+), \tag{2.6}$$

$$u^\varepsilon(x_-, t) = u_-(t); \quad u^\varepsilon(x_+, t) = u_+(t) \quad \text{for } t \in \mathbf{R}^+ \tag{2.7}$$

converge a.e. to a function u belonging to BV space as ε goes to zero and that the limit function u satisfies the boundary condition (2.3) in the weak sense of (2.3'). Also they proved the uniqueness of weak entropy solution for (2.1), (2.2) and (2.3'). Further LeRoux in an unpublished work [15] considered finite difference methods for (2.1)–(2.3) and proved that numerical solutions converge to a function satisfying the same boundary condition (2.3'). There are several papers [9,17,19] concerned with the initial-boundary value problems of conservation laws by using the vanishing viscosity methods, finite difference schemes and finite element methods.

When f is strictly convex i.e., when $f'' > 0$, the boundary conditions (2.3') are equivalent to

$$\text{either } u(x_{\pm}, t) = \bar{u}_{\pm}(t) \text{ or } \pm f'(u(x_{\pm}, t)) \geq 0 \text{ and } \pm f(u(x_{\pm}, t)) \leq \pm f(\bar{u}_{\pm}(t)),$$

where $\bar{u}_{\pm}(t) = \max\{u_{\pm}(t), \bar{u}\}$ with $\bar{u} = \inf_u f(u)$.

3. Artificial boundary conditions and finite difference schemes

In order to compute the initial value problems of (1.1) and (1.2) numerically, we need to introduce two artificial boundaries, say for example $x = x_-$ and $x = x_+$ with $-\infty < x_- < x_+ < \infty$, which contain a finite computing domain, and two artificial boundary conditions at the two boundaries. From the well-posedness of the initial-boundary problems given in the last section we see that two boundary values of u at the artificial boundaries are needed to complete the proposal for the artificial boundary conditions. By using the Lax's explicit formulas to the initial value problems of convex conservation laws we can calculate the boundary values of u at the artificial boundaries $x = x_{\pm}$ explicitly.

As observed by Lax [13,14] the exact formulas for $u(x, t)$ can be obtained for the initial value problem of convex conservation laws

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0 \quad \text{for } (x, t) \in \mathbf{R} \times \mathbf{R}^+, \quad (3.1)$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \mathbf{R}, \quad (3.2)$$

where $f(u)$ is strictly convex, that is $f''(u) > 0$. The solution is constructed as follows. Since f is convex, f' is increasing and it has inverse $(f')^{-1}$. Let g be a primitive function of the inverse of f' i.e.,

$$g(y) = \int^y (f')^{-1}(\eta) d\eta \quad (3.3)$$

and $F(x, t, y)$ be

$$F(x, t, y) = tg\left(\frac{x-y}{t}\right) + \int_0^y u_0(\xi) d\xi, \quad (3.4)$$

where the initial data u_0 is subject to

$$F(x, t, y) \rightarrow \infty \quad \text{for } |y| \rightarrow \infty, \quad (3.5)$$

then the entropy solution u to the initial value problem (3.1) and (3.2) is explicitly expressed by

$$u(x, t) = (f')^{-1}\left(\frac{x-y(x, t)}{t}\right) \quad (3.6)$$

for all (x, t) such that $F(x, t, y)$ has a unique minimum point $y(x, t)$. This is true except in a set which is countable for fixed t , and u is continuous in (x, t) , where the minimum point is unique. In general

$$u(x \pm 0, t) = (f')^{-1}\left(\frac{x - y_{\pm}(x, t)}{t}\right),$$

where $y_{\pm}(x, t)$ is the largest (smallest) minimum point of $F(x, t, y)$.

Here we have to point out that for the famous Burgers' equation, the flux is

$$f(u) = u^2/2,$$

$f'(u) = u$ and $(f')^{-1}(\eta) = \eta$. Therefore u is defined by

$$u(x, t) = \frac{x - y(x, t)}{t} \quad (3.7)$$

and $F(x, t, y)$ has the simple form

$$F(x, t, y) = \frac{(x-y)^2}{2t} + \int_0^y u_0(\xi) d\xi. \quad (3.8)$$

It is easy to show that if u_0 satisfies

$$\int_0^y u_0(\xi) d\xi = o(y^2),$$

then (3.5) holds.

The above statements are proved by Hopf [8] for the Burgers' equation and by Lax [13,14] for the convex conservation laws.

By using the explicit formula (3.6) for the initial value problem we can get the boundary values of u at the artificial boundaries $x = x_{\pm}$:

$$u(x_{\pm}, t) = (f')^{-1}\left(\frac{x_{\pm} - y(x_{\pm}, t)}{t}\right), \quad (3.9)$$

where $y(x, t)$ is the unique minimum of $F(x, t, y)$ for almost all (x, t) . Thus we have now completed the proposal for the artificial boundary conditions, i.e., we have formulated the following initial-boundary value problem:

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0 \quad \text{for } (x, t) \in (x_-, x_+) \times \mathbf{R}^+, \quad (3.10)$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in (x_-, x_+), \quad (3.11)$$

$$u(x_-, t) = u_-^{art}(t); \quad u(x_+, t) = u_+^{art}(t) \quad \text{for } t \in \mathbf{R}^+, \quad (3.12)$$

where the artificial boundary values $u_{\pm}^{art}(t)$ are defined by (3.9):

$$u_{\pm}^{art}(t) = (f')^{-1}\left(\frac{x_{\pm} - y(x_{\pm}, t)}{t}\right). \tag{3.13}$$

Now we consider the following three-point monotone difference schemes

$$v_j^{n+1} = v_j^n - \lambda(\bar{f}(v_j^n, v_{j+1}^n) - \bar{f}(v_{j-1}^n, v_j^n)) \tag{3.14}$$

applied to the conservation laws (3.10), where the numerical flux function $\bar{f}(u, v)$ satisfies the consistence condition

$$\bar{f}(v, v) = f(v)$$

and the right hand of the difference schemes i.e.,

$$H(u, v, w) \equiv v - \lambda(\bar{f}(v, w) - \bar{f}(u, v))$$

satisfies the monotone conditions: $H(u, v, w)$ is a nondecreasing function of each of its arguments, where $\lambda = \Delta t/\Delta x$ and Δt and Δx are time and space steps, respectively. Here v_j^n are the numerical solutions, which approximate the exact solution $u(x, t)$ at the point (x_j, t_n) , where $x_j = j\Delta x$ for $j = 0, \pm 1, \pm 2, \dots$ and $t_n = n\Delta t$ for $n = 0, 1, 2, \dots$ are space and time grid points, respectively. The monotone difference schemes are of first order accuracy [6] and include several popular schemes such as Lax–Friedrichs scheme [12], Godunov scheme [4] and Engquist–Osher scheme [3]. It is well known that the monotone schemes approach BV-discontinuous solutions for the initial value problem of the scalar convection equation at a rate only half in the L^1 -norm [11,20,18] and for the convex conservation laws the monotone schemes approximate piecewise constant solutions with finitely many shock discontinuities at rate one in L^1 -norm [22].

For the initial-boundary value problem (3.10)–(3.12) we assume that $x_- = L_- \Delta x$ and $x_+ = L_+ \Delta x$, where L_{\pm} are two integers and $\Delta x = (x_+ - x_-)/(L_+ - L_-)$. The monotone schemes assume the average initial value as follows:

$$v_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(\xi) d\xi \quad \text{for } j = L_-, L_- + 1, \dots, L_+, \tag{3.15}$$

where $x_{j\pm 1/2} = (j \pm 1/2)\Delta x$. Since the monotone difference schemes are quit similar to the viscosity methods (2.5)–(2.7), the monotone schemes also assume the average boundary values in the strong sense

$$v_{L_{\pm}}^n = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} u_{\pm}^{art}(\tau) d\tau \quad \text{for } n = 1, 2, \dots \tag{3.16}$$

In most of numerical tests the discontinuous solutions are piecewise smooth functions with only finitely many discontinuities. For this kind of solutions, v_j^n can assume the point-wise values, i.e.,

$$v_j^0 = u_0(x_j) \tag{3.17}$$

and

$$v_{L_{\pm}}^n = u_{\pm}^{art}(t_n). \tag{3.18}$$

With the aid of the initial-boundary conditions (3.15) and (3.16) or (3.17) and (3.18) we can solve the difference schemes (3.14) with $j = L_- + 1, \dots, L_+ - 1$ from $n = 1, 2, \dots$.

Remark. We have to point out that the numerical schemes assume the boundary values (3.16) or (3.18) at both of the artificial boundaries and we don't need to care if x_{\pm} is inflow or outflow boundary. The convergence theorems of difference schemes given in [15,9] make sure that the limit function of the difference schemes satisfies the boundary conditions in the weak sense (2.3').

Of course the artificial boundary conditions (3.16) or (3.18) can be used by any other numerical schemes.

4. Computing the entropy solution by using the explicit formulas

In this section we will consider that the entropy solution $u(x, t)$ to the initial value problem of convex conservation laws (1.1) and (1.2) is solved numerically by using the explicit formulas (3.3), (3.4) and (3.6).

As described in the previous section the entropy solution to the initial value problem (1.1) and (1.2) is explicitly expressed by

$$u(x, t) = (f')^{-1}\left(\frac{x - y(x, t)}{t}\right), \tag{4.1}$$

where $y(x, t)$ is the unique minimum point of $F(x, t, y)$ for almost all (x, t) and $F(x, t, y)$ is defined by

$$F(x, t, y) = tg\left(\frac{x - y}{t}\right) + \int_0^y u_0(\xi) d\xi, \tag{4.2}$$

with

$$g(y) = \int^y (f')^{-1}(\eta) d\eta.$$

In order to find $y(x, t)$ we can solve the problem

$$\min_y F(x, t, y) \quad (4.3)$$

numerically by using the function $y = \text{fminbnd}(fun, y_1, y_2)$ from MATLAB, where the function returns a value y that is a local minimizer of the function which is described in fun in the interval $y_1 < y < y_2$. Since y_1 and y_2 have to be provided in the MATLAB program, we will give a possible interval $[y_1, y_2]$. It follows from the expression (4.1) that $y(x, t)$ satisfies

$$y(x, t) = x - ta(u(x, t)), \quad (4.4)$$

where

$$a(u) = f'(u).$$

It is well known that the entropy solution $u(x, t)$ to the initial value problem (1.1) and (1.2) is bounded by the initial data

$$\inf_x u_0(x) \leq u(x, t) \leq \sup_x u_0(x)$$

and hence, on account of (4.4) and (1.3), $y(x, t)$ is bounded by

$$x - ta(\sup_x u_0(x)) \leq y(x, t) \leq x - ta(\inf_x u_0(x)).$$

Therefore we can choose

$$y_1 = x - ta(\sup_x u_0(x)) - M \quad \text{and} \quad y_2 = x - ta(\inf_x u_0(x)) + M, \quad (4.5)$$

where M is some positive constant. For given (x, t) the minimizer $y(x, t)$ of $F(x, t, y)$ can be solved numerically by the function “fminbnd” with the parameters y_1 and y_2 defined by (4.5) and then $u(x, t)$ is obtained by (4.1).

Remark. Since “fminbnd” can only solve for a local minimizer of $F(x, t, y)$, sometimes we have to divide the interval $[y_1, y_2]$ into two or three subintervals, find a local minimizer on each subinterval and choose the global minimizer from those local minimizers.

Remark. We make some comments on the accuracy of the min- F procedure. The termination tolerance “TolX” on y for the function “fminbnd” is an optional parameter. It follows from (4.1) that a tolerance TolX on $y(x, t)$ induces an error in $u(x, t)$, which is $O(\text{TolX}/t)$. In order to match with the numerical scheme used in the computational domain we have to choose the tolerance to be compatible with the accuracy of the numerical scheme. More precisely, if the numerical scheme is of r th order accuracy, then the tolerance should be in proportion to the r th power of the mesh size, i.e., $\text{TolX} = O(\Delta x^r)t$ where Δx is the spatial mesh size. Since the Lax–Friedrichs scheme is a first order accurate scheme we choose $\text{TolX} = O(\Delta x)t$ in the numerical computations given in the next section.

Remark. If the initial data $u_0(x) \in C^1$, then there exists a smooth solution $u(x, t) \in C^1$ for $0 < t < T$, where T is defined by $1/T = \sup_x \{-u_0' f''(u_0)\}$, and the characteristic method gives the relationship

$$u - u_0(x - ta(u)) = 0, \quad (4.6)$$

where $a(u) = f'(u)$. Therefore for smooth u_0 we can use (4.6) to obtain the local smooth solution $u(x, t)$ for $0 < t < T$. Harten et al. [5] use the relationship (4.6) to compute the local smooth solution to the Burgers' equation with $u_0(x) = \sin(\pi x)$ and use the smooth solution as boundary data in numerical experiments.

Notice that the solver given by (4.1) and (4.2) can compute the entropy solutions globally for $0 < t < \infty$. We will use the solver in next section to computing both the artificial boundary values and the entropy solutions of the examples.

5. Numerical examples

In this section we will compute two numerical examples, where the initial data are periodic or having a compact support, by using the proposed artificial boundary conditions. We also consider an example, of which the initial data are neither periodic nor having a compact support. There is no any numerical result available for this kind of initial value problems, but our proposed artificial boundary conditions can be used for those problems. In all of the numerical tests we simply use the Lax–Friedrichs monotone difference scheme:

$$v_j^{n+1} = \frac{v_{j+1}^n + v_{j-1}^n}{2} - \frac{\lambda}{2} (f(v_{j+1}^n) - f(v_{j-1}^n)); \quad j = L_- + 1, \dots, L_+ - 1; \tag{5.1}$$

$$v_{L_-}^n = v^{art}(t_n); \quad v_{L_+}^n = u_+^{art}(t_n); \tag{5.2}$$

$$v_j^0 = u_0(x_j), \tag{5.3}$$

where $x_{\pm} = L_{\pm}\Delta x$ are artificial boundaries, $u_{\pm}^{art}(t)$ are artificial boundary values given by (3.13), $\Delta x = (x_+ - x_-)/(L_+ - L_-)$, $u_0(x)$ is the initial data and $\lambda = \Delta t/\Delta x$ satisfies Courant–Friedrichs–Lewy condition

$$\lambda < 1 / \sup_x |f'(u_0(x))|. \tag{5.4}$$

It is known that under the stable condition (5.4) the Lax–Friedrichs scheme (5.1) is a monotone difference scheme with a numerical flux \bar{f} given by

$$\bar{f}(u, v) = \frac{f(u) + f(v)}{2} - \frac{1}{2\lambda}(u - v).$$

The explicit difference scheme (5.1) with the boundary conditions (5.2) and initial condition (5.3) can be solved from $n = 1, 2, \dots$

Of course, any more accurate and efficient numerical methods can be used with the proposed artificial boundary conditions.

Example 1. We consider the initial value problem of Burgers' equation

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = 0, \tag{5.5}$$

$$u(x, 0) = u_0(x), \tag{5.6}$$

where u_0 is a period function

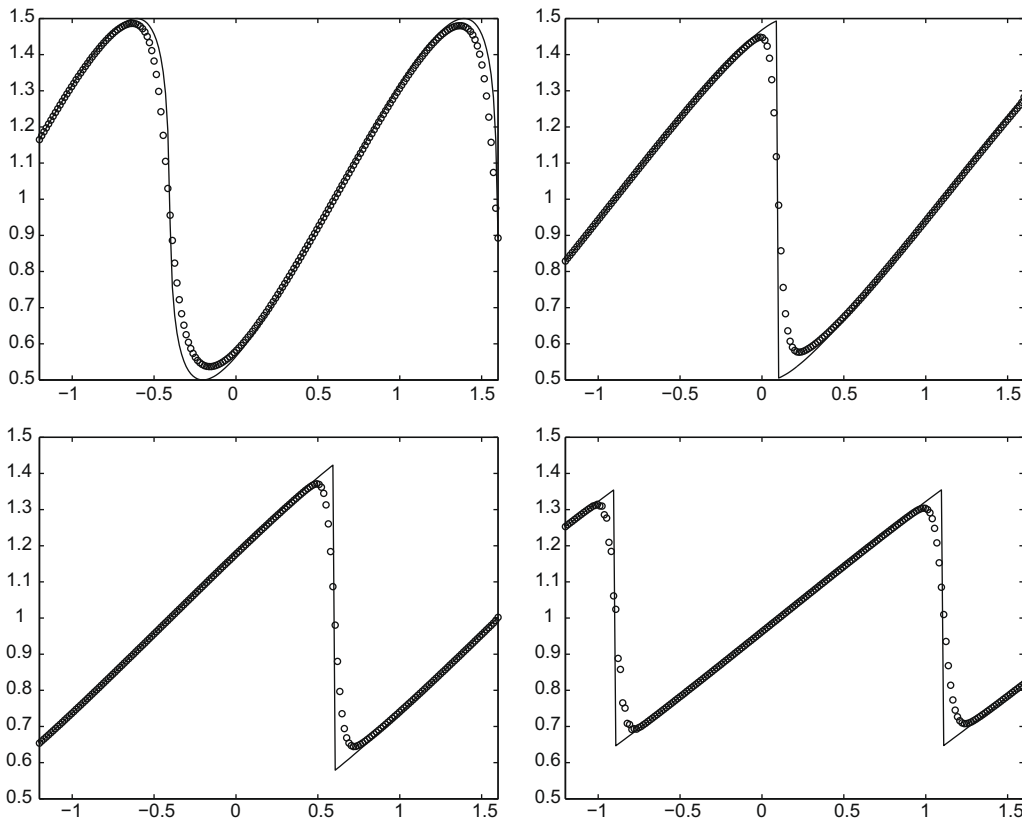


Fig. 1. Numerical “o” and exact “—” solution to (5.5) and (5.7) with $\Delta x = 2.8/200$ at times $t = 0.6, t = 1.1, t = 1.6$ and $t = 2.1$. There is no any spurious wave reflected from the artificial boundaries $x_- = -1.2$ and $x_+ = 1.6$.

$$u_0(x) = 1 + \frac{1}{2} \sin(\pi x), \quad (5.7)$$

with period $x \in [-1, 1)$.

This is a widely used example to test numerical methods by using periodic conditions, where $u(-1, t) = u(1, t)$ is specified, and the computational domain is restricted within $[-1, 1]$. Since our exact artificial boundary conditions can be used to any prescribed artificial boundaries, i.e., $x = x_{\pm}$, we may choose $x_- = -1.2$ and $x_+ = 1.6$, where $[-1.2, 1.6]$ is not a period for the initial data. It follows from the expression (4.2) and the initial data (5.7) that

$$F(x, t, y) = \frac{(x-y)^2}{t} + \left(y - \frac{\cos(\pi y) - 1}{2\pi} \right)$$

and the artificial boundary values at $x = -1.2$ and $x = 1.6$ are given by

$$u_-^{art}(t) = \frac{-1.2 - y(-1.2, t)}{t} \quad \text{and} \quad u_+^{art}(t) = \frac{1.6 - y(1.6, t)}{t},$$

where $y(-1.2, t)$ and $y(1.6, t)$ are solved from (4.3). Fig. 1 shows the numerical solution to the initial value problem (5.5) and (5.7) computed with the Lax–Friedrichs scheme (5.1), (5.2) and (5.3), and the exact solution is computed with the explicit formulas (3.7), (3.8) and (4.3). Here $\Delta x = 2.8/200$, $\lambda = \Delta t/\Delta x = 0.5$. From the figure we can see that a shock wave crosses the left artificial boundary $x = -1.2$ in between $t = 1.6$ and $t = 2.1$, but no any numerical oscillation occurs at the artificial boundary.

Example 2. The second example is the initial value problem (5.5) and (5.6) with an initial data u_0 having a compact support:

$$u_0(x) = \begin{cases} 1, & \text{if } x \in (0.5, 1.5), \\ 0, & \text{otherwise.} \end{cases} \quad (5.8)$$

Since the initial data u_0 has a compact support $(0.5, 1.5)$, the zero solution will not be disturbed outside of the region, which is included by two straight lines $x = 0.5 + t \min_x u_0(x)$ and $x = 1.5 + t \max_x u_0(x)$ for $t \geq 0$. In the traditional numerical methods two artificial boundaries are set outside of this region and zero boundary conditions are applied to the two boundaries. For large time t , we have to choose two artificial boundaries far apart and to calculate the difference schemes on a very large computational domain. Our proposed artificial boundaries will not be subject to this restriction and can be set at any places. In this example we set $x_- = 0$ and $x_+ = 2$. It follows from the expression (4.2) and the initial data (5.8) that

$$F(x, t, y) = \frac{(x-y)^2}{2t} + \begin{cases} 0, & y \leq 0.5, \\ y - 0.5, & y \in (0.5, 1.5), \\ 1, & y \geq 1.5 \end{cases}$$

and the artificial boundary values at $x = 0$ and $x = 2$ are given by

$$u_-^{art}(t) = \frac{0 - y(0, t)}{t} \quad \text{and} \quad u_+^{art}(t) = \frac{2 - y(2, t)}{t},$$

where $y(0, t)$ and $y(2, t)$ are solved from (4.3). Fig. 2 shows the numerical solution to the initial value problem (5.5) and (5.8) computed with the Lax–Friedrichs scheme (5.1), (5.2) and (5.3), and the exact solution is computed with the explicit formulas (3.7), (3.8) and (4.3). In the numerical computations $\lambda = 0.75$ and $\Delta x = 2/180$. From the figure we can see that there is no spurious numerical oscillation occurs at the artificial boundaries $x_- = 0$ and $x_+ = 2$.

In order to test the L^1 -convergence rate we compute Examples 1 and 2 by using refining meshes and the results are shown on Table 1, where l^1 -error is defined by

$$\|v_{\Delta x}(\cdot, t_n) - u(\cdot, t_n)\|_{l^1} = \sum_{j=L}^{L_+} |v_j^n - u(x_j, t_n)| \Delta x,$$

with $x_{\pm} = L_{\pm} \Delta x$. From the table we see that the convergence rates for both Examples 1 and 2 approach one and the conclusion coincides with the theoretical analysis given in [22,21,17].

Example 3. The third example is the initial value problem (5.5) and (5.6) with an initial data u_0 defined by

$$u_0(x) = (1 + x) \sin(\pi x), \quad (5.9)$$

which is neither periodic nor having a compact support.

Since the initial data u_0 is neither periodic nor having a compact support, so far there is no numerical computation available for this kind of initial value problems. But our artificial boundary conditions can deal with this kind of problems and we can set the two artificial boundaries at any locations, say for example $x_- = -1$ and $x_+ = 1.6$.

It follows from the expression (4.2) and the initial data (5.9) that

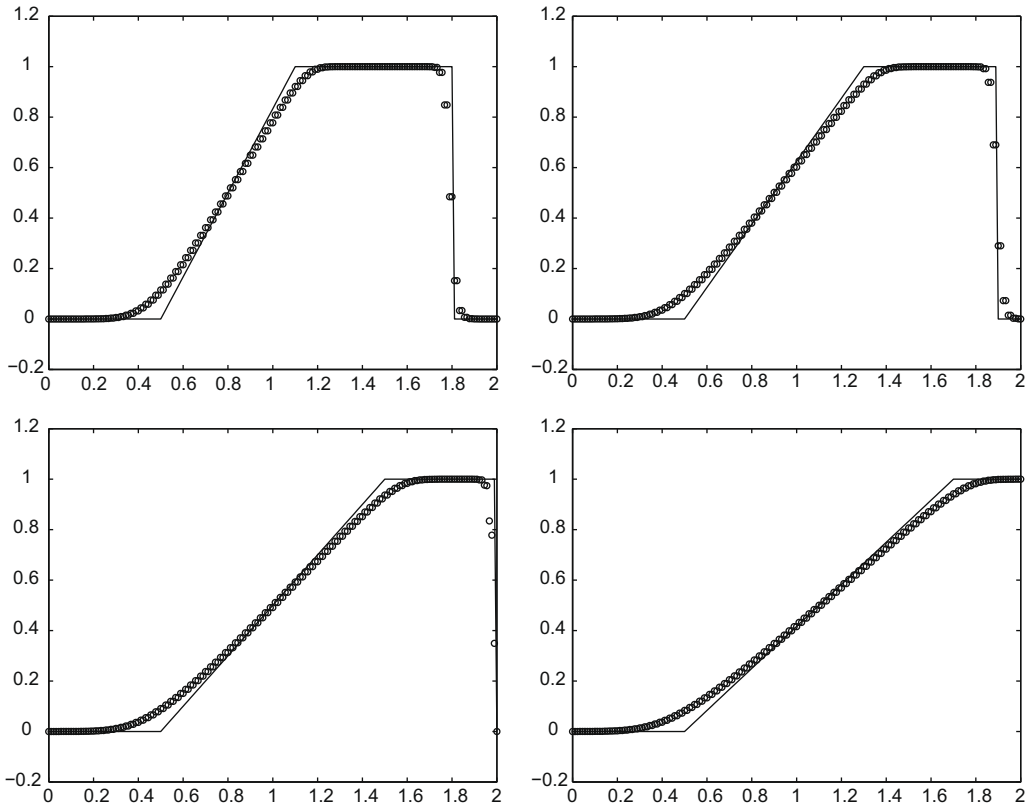


Fig. 2. Numerical “o” and exact “—” solution to (5.5) and (5.8) with $\Delta x = 2/180$ at times $t = 0.6, t = 0.8, t = 1.0$ and $t = 1.2$. There no any spurious numerical oscillation occurs at the artificial boundaries $x_- = 0$ and $x_+ = 2$.

Table 1

l^1 -Errors and convergence rates for the numerical solutions $v_{\Delta s}$.

$t = 1.1$	Example 1		$t = 0.8$	Example 2	
Δx	l^1 -Error	l^1 -Rate	Δx	l^1 -Error	l^1 -Rate
$0.28/2^2$	0.2108	–	$0.10/2^3$	0.0658	–
$0.28/2^3$	0.1200	0.8129	$0.10/2^4$	0.0362	0.8607
$0.26/2^4$	0.0623	0.9462	$0.10/2^5$	0.0212	0.7759
$0.28/2^5$	0.0300	1.0544	$0.10/2^6$	0.0115	0.8780
$0.28/2^6$	0.0147	1.0329	$0.10/2^7$	0.0065	0.8143
$0.28/2^7$	0.0073	1.0024	$0.10/2^8$	0.0035	0.8972
$0.28/2^8$	0.0036	1.0240	$0.10/2^9$	0.0020	0.8460
$0.28/2^9$	0.0016	0.9978	$0.10/2^{10}$	0.0010	0.9163

$$F(x, t, y) = \frac{(x - y)^2}{2t} + \left(\frac{1 - (1 + y) \cos(\pi y)}{\pi} + \frac{\sin(\pi y)}{\pi^2} \right)$$

and the artificial boundary values at $x = -1$ and $x = 1.6$ are given by

$$u_-^{art}(t) = \frac{-1 - y(-1, t)}{t} \quad \text{and} \quad u_+^{art}(t) = \frac{1.6 - y(1.6, t)}{t},$$

where $y(-1, t)$ and $y(1.6, t)$ are solved from (4.3). Fig. 3 shows the numerical solution to the initial value problem (5.5) and (5.9) computed with the Lax–Friedrichs scheme (5.1), (5.2) and (5.3), and the exact solution is computed with the explicit formulas (3.7), (3.8) and (4.3). In the numerical computations $\lambda = 0.40$ and $\Delta x = 2.6/150$. From the figure we can see that the artificial boundaries $x_- = -1.0$ and $x_+ = 1.6$ do not give rise to any spurious wave reflection.

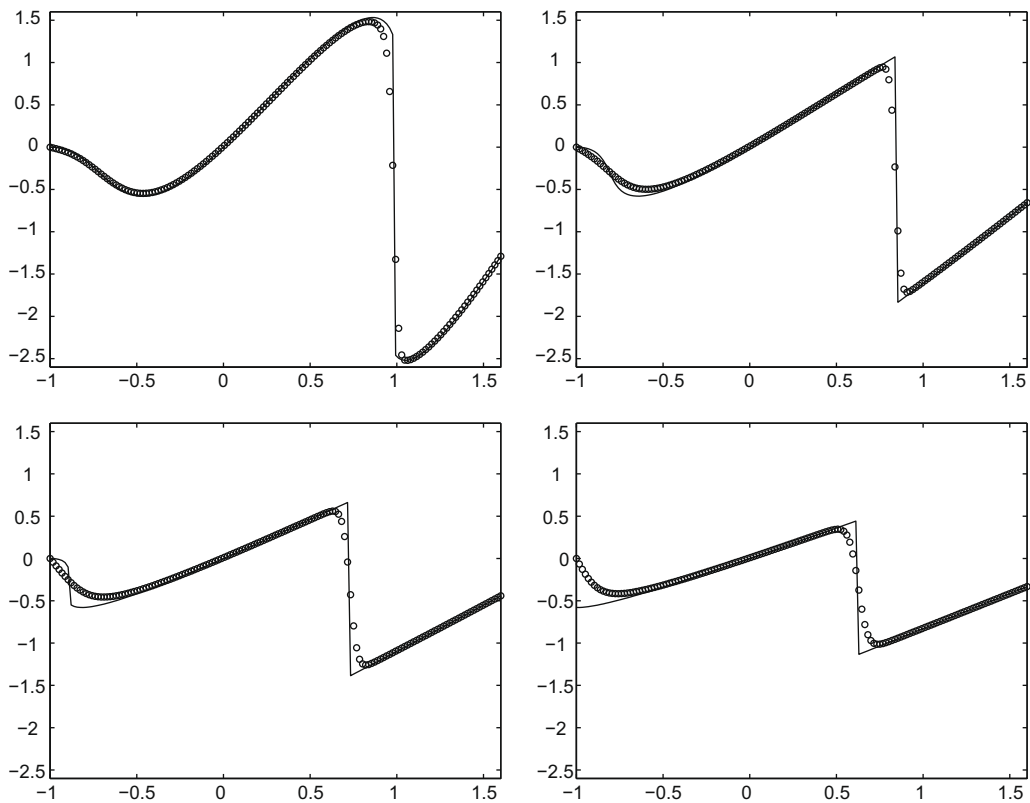


Fig. 3. Numerical “o” and exact “-” solution to (5.5) and (5.9) with $\Delta x = 2.6/150$ at times $t = 0.2$, $t = 0.5$, $t = 0.8$ and $t = 1.1$. There the artificial boundaries $x_- = -1.0$ and $x_+ = 1.6$ do not give rise to any spurious wave reflection.

6. Conclusions

In this paper we propose an artificial boundary condition for computing initial value problems of convex conservation laws. The approach adopted is to apply the exact solution on the artificial boundaries. For scalar convex conservation laws, such a solution can be obtained by the Lax-formulas, a minimization of a functional involving flux function and primitive of the initial data, and the solution can be calculated using a MATLAB routine.

Over years some efforts are made to extend Lax-formulas to the case with non-convex flux. For example, Kunik [10] proves a solution formula for a non-convex scalar hyperbolic conservation law with monotone initial data. Therefore it is easy to extend the result given in this paper to the non-convex case.

Acknowledgment

The author thanks an anonymous referee for valuable comments and suggestions.

References

- [1] C. Bardos, A.Y. LeRoux, J.C. Nedelec, First order quasilinear equations with boundary conditions, *Comm. Partial Differen. Equat.* 4 (1979) 1017–1034.
- [2] B. Cockburn, C.W. Shu, TVB Runge–Kutta local projecting discontinuous Galerkin finite element methods for conservation laws II: general framework, *Math. Comput.* 52 (1989) 411–435.
- [3] B. Engquist, S. Osher, One-sided difference approximations for nonlinear conservation laws, *Math. Comput.* 36 (1981) 321–351.
- [4] S.K. Godunov, Finite difference methods for numerical computations of discontinuous solutions of equations of fluid dynamics, *Mat. Sb.* 47 (1959) 259–271.
- [5] A. Harten, B. Engquist, S. Osher, S.R. Chakravarthy, Uniformly high order accurate essentially non-oscillatory schemes, III, *J. Comput. Phys.* 71 (1987) 231–303.
- [6] A. Harten, J.M. Hyman, P.D. Lax, On finite-difference approximation and entropy conditions for shocks, *Commun. Pure Appl. Math.* 29 (1976) 297–322.
- [7] A. Harten, S. Osher, Uniformly high-order accurate nonoscillatory schemes, I, *SIAM J. Numer. Anal.* 24 (1987) 279–309.
- [8] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Commun. Pure Appl. Math.* 3 (1950) 201–230.
- [9] K.T. Joseph, P.G. LeFloch, Boundary layers in weak solutions of hyperbolic conservation laws, *Arch. Rational Mech. Anal.* 147 (1999) 47–88.
- [10] M. Kunik, A solution formula for a non-convex scalar hyperbolic conservation law with monotone initial data, *Math. Meth. Appl. Sci.* 16 (1993) 895–902.

- [11] N.N. Kuznetsov, Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation, *USSR Comput. Math. Math. Phys.* 16 (1976) 105–119.
- [12] P.D. Lax, Weak solutions of nonlinear hyperbolic equations and their numerical computation, *Commun. Pure Appl. Math.* 7 (1954) 159–193.
- [13] P.D. Lax, Hyperbolic systems of conservation laws II, *Commun. Pure Appl. Math.* 10 (1957) 537–566.
- [14] P.D. Lax, Hyperbolic systems of conservation laws and the mathematical theory of shock waves, *Regional Conf. Appl. Math.* 11 (1973) 1–48.
- [15] A.Y. LeRoux, Approximation de quelques problèmes hyperboliques nonlinéaires, Thèse d'état, University of Rennes, France, 1979.
- [16] R.J. LeVeque, *Numerical Methods for Conservation Laws*, Birkhäuser Verlag, 1990.
- [17] H.X. Liu, T. Pan, Construction of solutions and L^1 -error estimates of viscous methods for scalar conservation laws with boundary, *Acta Math. Sinica* 23 (2007) 393–410.
- [18] F. Sabac, The optimal convergence rate of monotone difference methods for hyperbolic conservation laws, *SIAM J. Numer. Anal.* 34 (1997) 2306–2318.
- [19] A. Szepessy, Measure-valued solutions to scalar conservation laws with boundary conditions, *Arch. Rational Mech. Anal.* 107 (1989) 181–193.
- [20] T. Tang, Z.H. Teng, The sharpness of Kuznetsov's $O(\sqrt{\Delta x})$ L^1 -error estimate for monotone difference schemes, *Math. Comput.* 64 (1995) 581–589.
- [21] T. Tang, Z.H. Teng, Viscosity methods for piecewise smooth solutions to scalar conservation laws, *Math. Comput.* 66 (1997) 495–526.
- [22] Z.H. Teng, P.W. Zhang, Optimal L^1 -rate of convergence for viscosity method and monotone scheme to piecewise constant solutions with shocks, *SIAM J. Numer. Anal.* 34 (1997) 959–978.